On continuous *q*-Hermite polynomials and the classical Fourier transform

2008 J. Phys. A: Math. Theor. 41 125201

(http://iopscience.iop.org/1751-8121/41/12/125201)

J. Phys. A: Math. Theor. **41** (2008) 125201 (9pp) doi:10.1088/1751-8113/41/12/125201

On continuous q-Hermite polynomials and the classical Fourier transform

M K Atakishiyeva[1] **and N M Atakishiyev**[2]

[1] Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, C.P. 62250 Cuernavaca, Morelos, Mexico
[2] Instituto de Matemáticas, Universidad Nacional Autónoma de México, C.P. 62210 Cuernavaca, Morelos, Mexico

Received 9 September 2007, in final form 6 February 2008
Published 10 March 2008
Online at stacks.iop.org/JPhysA/41/125201

Abstract
We prove that the classical Fourier-transform operator $\widehat{\mathcal{F}}$ intertwines two q-difference equations for the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers, which are associated with the two distinct sets of values for the parameter q: $0 < q < 1$ and $1 < q < \infty$.

PACS numbers: 02.30.Nw, 02.30.Gp

1. Introduction

It is well known that an explicit realization of the Macfarlane–Biedenharn q-oscillator [1, 2] can be formulated in terms of the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers [3]. The continuous q-Hermite polynomials $H_n(x|q)$ are also closely related to the oscillator representations of the quantum algebra $su_q(1, 1)$ [4, 5].

This paper started from an attempt to make transparent an algebraic structure of q-difference equations for the q-Hermite polynomials $H_n(x|q)$, which originates such striking harmony of these polynomials with the q-independent classical Fourier integral transform. Having chosen the continuous q-Hermite polynomials $H_n(x|q)$ as a beginning for our study, we show in what follows that the Fourier transform intertwines two q-difference equations for the $H_n(x|q)$, which correspond to the distinct set of values for the parameter $q : 0 < q < 1$ and $1 < q < \infty$.

Let us briefly recall some mathematical aspects of the classical Fourier integral transform. The orthonormalized wavefunctions $\psi_n(x)$ of the linear harmonic oscillator in non-relativistic quantum mechanics,

$$\int_{\mathbb{R}} \psi_m(x)\psi_n(x)\,\mathrm{d}x = \delta_{mn}, \qquad m, n = 0, 1, 2, \ldots, \tag{1.1}$$

are explicitly given as

$$\psi_n(x) := c_n H_n(\xi)\exp(-\xi^2/2), \qquad 1/c_n = \sqrt{\sqrt{\pi}2^n n!}, \tag{1.2}$$

where $H_n(\xi)$ are the classical Hermite polynomials and $\xi := \sqrt{m\omega/\hbar}\,x$ is a dimensionless coordinate (see, for example, [6]). In quantum mechanics they emerge as eigenfunctions of the Hamiltonian $\widehat{H}$ for the linear harmonic oscillator,

$$\widehat{H}\,\psi_n(x) \equiv \frac{\hbar\omega}{2}\left(\xi^2 - \frac{d^2}{d\xi^2}\right)\psi_n(x) = \hbar\omega\,(n+1/2)\,\psi_n(x), \tag{1.3}$$

which is a self-adjoint differential operator of the second order. The linear harmonic oscillator wavefunctions $\psi_n(x)$ (we recall that the functions $H_n(y)\,e^{-y^2/2}$ are usually referred to as *Hermite functions* in the mathematical literature) represent an important explicit example of an orthonormal and complete system in the Hilbert space $L^2(\mathbb{R}, dx)$ of square-integrable functions on the full real line $x \in \mathbb{R}$. It is further well known that the wavefunctions of the linear harmonic oscillator $\psi_n(x)$ possess the simple transformation property with respect to the classical Fourier transform: *they are also eigenfunctions of the Fourier integral transform, associated with the eigenvalues* i^n, that is,

$$(\mathcal{F}\psi_n)(x) \equiv \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ixy}\,\psi_n(y)\,dy = i^n\,\psi_n(x). \tag{1.4}$$

Evidently, if one considers some nontrivial one-parameter extension of the system of functions $\psi_n(x)$, then an extended system will not have the same properties as the initial one, referred to above. For instance, it is natural that q-extensions of the Hermite polynomials do not reveal the same transformation property as the wavefunctions $\psi_n(x)$ do in (1.4). But what is rather surprising that *all known q-extensions* of the Hermite polynomials $H_n(x)$ exhibit simple behavior with respect to the Fourier transform [7, 8].

A complementary motivation for studying this topic comes from the understanding that the Fourier transform is in fact intimately connected with the *finite (discrete) Fourier transform* [9, 10]. This circumstance enables one to establish that these q-extensions of the Hermite polynomials $H_n(x)$ enjoy simple transformation properties *also* with respect to the finite Fourier transform (see [11] and references therein). In particular, it was shown in [12] that one can construct a one-parameter family of q-extensions for eigenvectors of the finite Fourier transform in terms of the continuous q-Hermite polynomials of Rogers. These q-extended eigenvectors are of interest from the point of view of their applications in signal analysis [13, 14] and as foundation for finite models in quantum mechanics [15, 16] and optics [17, 18].

Our avowed interest in the continuous q-Hermite polynomials of Rogers is incited by the fact that this family exhibits remarkable symmetry properties. Due to these properties they are one of the main instruments for studying q-oscillators and their applications (see, for example, [1, 2, 4]). In particular, it turns out that q-difference equations for this family are closely related to Hamiltonians of the associated systems. The continuous q-Hermite polynomials are also very useful as they provide an explicit simple realization of the oscillator representations of the quantum algebra $su_q(1,1)$, which are constructed in terms of the creation and annihilation operators for the q-oscillator [4, 5].

The layout of the paper is as follows. Section 2 collects those known facts about the continuous q-Hermite polynomials $H_n(x|q)$, which are needed for deriving an operator form of the Fourier integral transform for the q-Hermite polynomials in section 3 and an intertwining relation between two q-difference equations for the $H_n(x|q)$ in section 4. Section 5 closes the paper with a few brief remarks, which outline some further research directions of interest.

Throughout this exposition we employ standard notations of the theory of special functions (see, for example, [19] or [20]).

2. The continuous q-Hermite polynomials

The continuous q-Hermite polynomials $H_n(y|q)$ are those q-extensions of the Hermite polynomials $H_n(x)$, which are generated by the three-term recurrence relation

$$H_{n+1}(y|q) = 2yH_n(y|q) - (1-q^n)H_{n-1}(y|q), \qquad 0 < q < 1, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

with the initial condition $H_0(y|q) = 1$. These polynomials were introduced in 1894 by Rogers [3], although their orthogonality property on the finite interval $y \in [-1, 1]$ was derived by Allaway [21] only in 1980. The explicit form of $H_n(y|q)$ is exhibited by their Fourier expansion

$$H_n(y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathrm{e}^{\mathrm{i}(n-2k)\theta}, \qquad y = \cos\theta, \tag{2.2}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is the q-binomial coefficient.

It should be recalled that to consider the continuous q-Hermite polynomials $H_n(y|q)$ for the values of the parameter q in the interval $[1, \infty)$, it is customary to introduce the so-called continuous q^{-1}-Hermite polynomials $h_n(y|q)$ as [22]

$$h_n(y|q) := \mathrm{i}^{-n} H_n(\mathrm{i}y|q^{-1}). \tag{2.3}$$

In view of the inversion formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{2.4}$$

for the q-binomial coefficient, from (2.2) and (2.3) one deduces that

$$h_n(\sinh\chi|q) = \sum_{k=0}^{n} (-1)^k q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathrm{e}^{(n-2k)\chi}. \tag{2.5}$$

It is important to observe that in order to be able to formulate adequately transformation properties of the q-Hermite and q^{-1}-Hermite polynomials $H_n(y|q)$ and $h_n(y|q)$ with respect to the Fourier transforms one has to employ the following nonconventional parametrization for their argument y:

$$\begin{aligned} &y = \cos\theta = \sin\kappa x, \qquad \theta = \frac{\pi}{2} - \kappa x, \qquad \text{when} \qquad 0 < q < 1; \\ &y = \sinh\chi = \sinh\kappa x, \qquad \chi = \kappa x, \qquad \text{when} \qquad 1 < q < \infty. \end{aligned} \tag{2.6}$$

In both cases above $\kappa := \sqrt{\ln q^{-1/2}}$ or, equivalently, $q := \mathrm{e}^{-2\kappa^2}$. Thus, formulae (2.2) and (2.5) are representable as expansions

$$H_n(\sin\kappa x|q) = \mathrm{i}^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \mathrm{e}^{\mathrm{i}(2k-n)\kappa x}, \tag{2.7}$$

$$h_n(\sinh\kappa x|q) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} \mathrm{e}^{(n-2k)\kappa x}, \tag{2.8}$$

which are building blocks of the rest of this work. In particular, it should be noted that (2.7) and (2.8) were key relations in establishing the Fourier integral transform [23]

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_n(\sin\kappa y|q)\, \mathrm{e}^{\mathrm{i}xy - y^2/2} \mathrm{d}y = \mathrm{i}^n q^{n^2/4} h_n(\sinh\kappa x|q)\, \mathrm{e}^{-x^2/2}, \tag{2.9}$$

which interrelates q-Hermite and q^{-1}-Hermite polynomials. Finally, observe that it is not hard to check, by using the recurrence relation (2.1) and definition (2.3), the validity of limit relations

$$\lim_{q\to 1^-} \kappa^n H_n(\sin \kappa x|q) = \lim_{q\to 1^-} \kappa^n h_n(\sinh \kappa x|q) = H_n(x).$$

Therefore it is obvious that the Fourier integral transform (2.9) reduces to the classical result (1.4) for the ordinary Hermite polynomials $H_n(x)$ in the limit as $q \to 1$.

3. Operator form of the Fourier transform

We begin this section with the remark that in foregoing formulae, related to the classical Fourier transform, it proves more convenient to employ the self-adjoint *number operator*

$$\widehat{N} := \frac{1}{2}\left(\xi^2 - 1 - \frac{d^2}{d\xi^2}\right), \tag{3.1}$$

rather than the Hamiltonian $\widehat{H}$, defined by (1.3). These two quantum-mechanical operators, $\widehat{N}$ and the Hamiltonian $\widehat{H}$, are interrelated in a simple way $\widehat{N} = \frac{1}{\hbar\omega}\widehat{H} - 1/2$. From (1.3) and (3.1) it is evident that

$$\widehat{N}\psi_n(x) = n\psi_n(x). \tag{3.2}$$

Besides, to simplify notations in what follows we will be employing x and y instead of dimensionless variable ξ.

It is well known that integral Fourier transform in the Hilbert space $L^2(\mathbb{R}, dx)$ of square-integrable functions $f(x)$, $x \in \mathbb{R}$, can be represented in the operator form as

$$\widehat{\mathcal{F}} := \exp\left(\frac{\pi i}{2}\widehat{N}\right). \tag{3.3}$$

Since the number operator $\widehat{N}$ is self-adjoint, from (3.3) it is clear that

$$\widehat{\mathcal{F}}^{\dagger} \equiv \widehat{\mathcal{F}}^{-1} = \exp\left(-\frac{\pi i}{2}\widehat{N}\right) \tag{3.4}$$

and, consequently, $\widehat{\mathcal{F}}^{\dagger}\widehat{\mathcal{F}} = \widehat{\mathcal{F}}\widehat{\mathcal{F}}^{\dagger} = I$, which means that $\widehat{\mathcal{F}}$ is a *unitary operator*.

One readily verifies (by using definition (3.1) and formula (3.2)) that $\widehat{\mathcal{F}}\psi_n(x) = i^n\psi_n(x)$, that is, the eigenfunctions and eigenvalues of the unitary operator $\widehat{\mathcal{F}}$ are the same as of the integral Fourier transform (2.9). Observe also that by definition (3.3) the operators $\widehat{\mathcal{F}}$ and $\widehat{N}$ commute, that is,

$$[\widehat{\mathcal{F}}, \widehat{N}] \equiv \widehat{\mathcal{F}}\widehat{N} - \widehat{N}\widehat{\mathcal{F}} = 0. \tag{3.5}$$

Thus, a compact form of (3.5),

$$\widehat{\mathcal{F}}\widehat{N}\widehat{\mathcal{F}}^{-1} = \widehat{N}, \tag{3.6}$$

represents a standard algebraic way of expressing the fact that the number operator $\widehat{N}$ and the Fourier-transform operator $\widehat{\mathcal{F}}$ have a common set of the eigenfunctions $\psi_n(x)$, $n = 0, 1, 2, \ldots$. Note also that substituting explicit form (3.1) of the number operator $\widehat{N}$ into (3.6) and canceling common constant factors on both sides, one arrives at the relation

$$\widehat{\mathcal{F}}\left[x^2 + \widehat{p}_x^2\right]\widehat{\mathcal{F}}^{-1} = x^2 + \widehat{p}_x^2, \qquad \widehat{p}_x := -i\frac{d}{dx}. \tag{3.7}$$

At the same time, as we show below (see (3.10) for $m = 2$), it is straightforward to verify that

$$\widehat{\mathcal{F}}x^2\widehat{\mathcal{F}}^{-1} = \widehat{p}_x^2. \tag{3.8}$$

Consequently, (3.7) simply reduces to

$$\widehat{\mathcal{F}}\widehat{p}_x^2\widehat{\mathcal{F}}^{-1} = x^2. \tag{3.9}$$

Thus, the invariance of the number operator $\widehat{N}$ with respect to the similarity transformation (3.6) is rooted in the fact that, first, the governing Hamiltonian $\widehat{H}$ in (1.3) contains the coordinate x and the momentum operator $\widehat{p}_x := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ in a symmetric way and, second, the x^2 and $\widehat{p}_x^2$ transform into each other under this similarity transformation by $\widehat{\mathcal{F}}$. As we shall see in the next section, these two similarity transformations (3.8) and (3.9) of x^2 and $\widehat{p}_x^2$ into each other turn out to be key relations also in the case of the q-Hermite polynomials at hand.

Because of the importance of the integral Fourier transform (2.9) for the q-Hermite polynomials, we derive in this section its operator form. We begin with the relation

$$\widehat{\mathcal{F}}x^m = \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\widehat{\mathcal{F}}, \tag{3.10}$$

where m is some positive integer. For $m = 1$ it is not hard to verify, by using the three-term recurrence relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ and the formula of differentiation $\frac{\mathrm{d}}{\mathrm{d}x}H_n(x) = 2nH_{n-1}(x)$ for the Hermite polynomials $H_n(x)$, that

$$\widehat{\mathcal{F}}x\psi_n(x) = \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)\widehat{\mathcal{F}}\psi_n(x) \tag{3.11}$$

for the Hermite functions $\psi_n(x)$ with arbitrary $n = 0, 1, 2, \ldots$. The completeness of the system of functions $\{\psi_n(x)\}_{n=0}^{\infty}$ in the Hilbert space $L^2(\mathbb{R}, \mathrm{d}x)$ of square-integrable functions (see pp.306–309 in [20] for a proof of this statement) permits one to extend relation (3.11) to an arbitrary function $f(x) \in L^2(\mathbb{R}, \mathrm{d}x)$. Consequently, $\widehat{\mathcal{F}}x = \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)\widehat{\mathcal{F}}$ (in quantum mechanics this relation is equivalent to the statement that momentum p_x in the coordinate x-realization is represented by the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$) and iterating this formula m times in succession, one arrives at (3.10).

Multiply now both sides of (3.10) by $(\mathrm{i}a)^m/m!$, $a \in \mathbb{R}$, and sum them with respect to the m from zero to infinity to obtain that

$$\widehat{\mathcal{F}}\,\mathrm{e}^{\mathrm{i}ax} = \mathrm{e}^{a\frac{\mathrm{d}}{\mathrm{d}x}}\widehat{\mathcal{F}}, \tag{3.12}$$

where $\exp\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right) \equiv \widehat{T}_a$ is the unitary operator of the finite shift (displacement) over a distance a, whose action on any function $f(x) \in L^2(\mathbb{R}, \mathrm{d}x)$ is given as $\widehat{T}_a f(x) = f(x+a)$ (cf p.66 in [6]).

We are now in a position to derive an operator form of the Fourier integral transform (2.9). Indeed, with the aid of the explicit representation for the continuous q-Hermite polynomials (2.7), one can evaluate, by using at the last step (2.8), that

$$\begin{aligned}
\widehat{\mathcal{F}}\left\{H_n(\sin\kappa x|q)\,\mathrm{e}^{-x^2/2}\right\} &= \mathrm{i}^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\widehat{\mathcal{F}}\left\{\mathrm{e}^{\mathrm{i}(n-2k)\kappa x}\,\mathrm{e}^{-x^2/2}\right\}\\
&= \mathrm{i}^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\mathrm{e}^{(n-2k)\kappa\frac{\mathrm{d}}{\mathrm{d}x}}\widehat{\mathcal{F}}\left\{\mathrm{e}^{-x^2/2}\right\}\\
&= \mathrm{i}^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\mathrm{e}^{-[x+(n-2k)\kappa]^2/2}\\
&= \mathrm{i}^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\mathrm{e}^{(2k-n)\kappa x-(n-2k)^2\kappa^2/2}\,\mathrm{e}^{-x^2/2}
\end{aligned}$$

$$= \mathrm{i}^n q^{n^2/4}\, \mathrm{e}^{-x^2/2} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}\, \mathrm{e}^{(n-2k)\kappa x}$$

$$= \mathrm{i}^n q^{n^2/4} h_n(\sinh \kappa x | q)\, \mathrm{e}^{-x^2/2}. \tag{3.13}$$

Thus the desired operator form of the integral Fourier transform (2.9) is

$$\widehat{\mathcal{F}}\left\{H_n(\sin \kappa x | q) \mathrm{e}^{-x^2/2}\right\} = \mathrm{i}^n q^{n^2/4} h_n(\sinh \kappa x | q)\, \mathrm{e}^{-x^2/2}. \tag{3.14}$$

It should be emphasized that (3.14) is just another form of stating the key result (2.9) for the case under investigation. But in the next section we show that it does enable one to employ the operational calculus techniques in order to prove that the similarity transformation by $\widehat{\mathcal{F}}$ intertwines two q-difference equations for the continuous q-Hermite polynomials of Rogers (2.7) and (2.8).

4. Similarity transformation by $\widehat{\mathcal{F}}$ converts q into q^{-1}

The continuous q-Hermite polynomials $H_n(\sin \kappa x | q)$ satisfy the q-difference equation

$$[\mathrm{e}^{\mathrm{i}\kappa x}\, \mathrm{e}^{-\mathrm{i}\kappa \partial_x} + \mathrm{e}^{-\mathrm{i}\kappa x}\, \mathrm{e}^{\mathrm{i}\kappa \partial_x}] H_n(\sin \kappa x | q) = 2q^{-n/2} \cos \kappa x\, H_n(\sin \kappa x | q), \tag{4.1}$$

where $\partial_x \equiv \frac{\mathrm{d}}{\mathrm{d}x}$. We recall that this q-difference equation is a direct consequence of Roger's linear generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(\sin \kappa x | q) = e_q(\mathrm{i}t\, \mathrm{e}^{-\mathrm{i}\kappa x}) e_q(-\mathrm{i}t\, \mathrm{e}^{\mathrm{i}\kappa x}), \qquad |t| < 1, \tag{4.2}$$

for the continuous q-Hermite polynomials $H_n(\sin \kappa x | q)$ (see, for example, [19] or [20]). The $e_q(z)$ in (4.2) denotes the q-exponential function, defined as

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}. \tag{4.3}$$

To see now how (4.1) emerges from (4.2), act on both sides of (4.2) by the difference operator $\mathrm{e}^{\mathrm{i}\kappa x} \mathrm{e}^{-\mathrm{i}\kappa \partial_x} + \mathrm{e}^{-\mathrm{i}\kappa x} \mathrm{e}^{\mathrm{i}\kappa \partial_x}$. This yields

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} [\mathrm{e}^{\mathrm{i}\kappa x}\, \mathrm{e}^{-\mathrm{i}\kappa \partial_x} + \mathrm{e}^{-\mathrm{i}\kappa x}\, \mathrm{e}^{\mathrm{i}\kappa \partial_x}] H_n(\sin \kappa x | q)$$

$$= \mathrm{e}^{\mathrm{i}\kappa x} e_q(\mathrm{i}t\, \mathrm{e}^{-\mathrm{i}\kappa(x - \mathrm{i}\kappa)}) e_q(-\mathrm{i}t\, \mathrm{e}^{\mathrm{i}\kappa(x-\mathrm{i}\kappa)})$$

$$+ \mathrm{e}^{-\mathrm{i}\kappa x} e_q(\mathrm{i}t\, \mathrm{e}^{-\mathrm{i}\kappa(x+\mathrm{i}\kappa)}) e_q(-\mathrm{i}t\, \mathrm{e}^{\mathrm{i}\kappa(x+\mathrm{i}\kappa)}). \tag{4.4}$$

Since $q = \mathrm{e}^{-2\kappa^2}$ by definition, the right-hand side of (4.4) may be written as

$$\mathrm{e}^{\mathrm{i}\kappa x} e_q(\mathrm{i}q^{1/2} t\, \mathrm{e}^{-\mathrm{i}\kappa x}) e_q(-\mathrm{i}q^{-1/2} t\, \mathrm{e}^{\mathrm{i}\kappa x}) + \mathrm{e}^{-\mathrm{i}\kappa x} e_q(\mathrm{i}q^{-1/2} t\, \mathrm{e}^{-\mathrm{i}\kappa(x+\mathrm{i}\kappa)}) e_q(-\mathrm{i}q^{1/2} t\, \mathrm{e}^{\mathrm{i}\kappa(x+\mathrm{i}\kappa)}). \tag{4.5}$$

Apply then the functional equation $e_q(qz) = (1-z)e_q(z)$ for the q-exponential function (4.3) to the first q-exponential factor in the first line of (4.5) and to the second q-exponential factor in the second line of (4.5) to obtain that

$$[\mathrm{e}^{\mathrm{i}\kappa x}(1 - \mathrm{i}q^{-1/2}t\,\mathrm{e}^{-\mathrm{i}\kappa x}) + \mathrm{e}^{-\mathrm{i}\kappa x}(1 + \mathrm{i}q^{-1/2}t\,\mathrm{e}^{\mathrm{i}\kappa x})]e_q(\mathrm{i}q^{-1/2}t\,\mathrm{e}^{-\mathrm{i}\kappa x})e_q(-\mathrm{i}q^{-1/2}t\,\mathrm{e}^{\mathrm{i}\kappa x})$$
$$= 2\cos\kappa x\, e_q(\mathrm{i}q^{-1/2}t\,\mathrm{e}^{-\mathrm{i}\kappa x})e_q(-\mathrm{i}q^{-1/2}t\,\mathrm{e}^{\mathrm{i}\kappa x})$$
$$= 2\cos\kappa x\sum_{n=0}^{\infty}\frac{(q^{-1/2}t)^n}{(q;q)_n}H_n(\sin\kappa x|q).$$

One thus arrives at the identity

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}[\mathrm{e}^{\mathrm{i}\kappa x}\,\mathrm{e}^{-\mathrm{i}\kappa\partial_x} + \mathrm{e}^{-\mathrm{i}\kappa x}\,\mathrm{e}^{\mathrm{i}\kappa\partial_x}]H_n(\sin\kappa x|q) = 2\cos\kappa x\sum_{n=0}^{\infty}\frac{(q^{-1/2}t)^n}{(q;q)_n}H_n(\sin\kappa x|q). \tag{4.6}$$

It remains only to equate coefficients of the equal powers of the parameter t on both sides of (4.6) to obtain difference equation (4.1).

In the case when the parameter q lies in the interval $[1,\infty)$, the q-difference equation (4.1) takes the form (valuable background material about these two equations (4.1) and (4.7) for the continuous q-Hermite and q^{-1}-Hermite polynomials, respectively, can be found in [24, 25])

$$[\mathrm{e}^{\kappa x}\,\mathrm{e}^{-\kappa\partial_x} + \mathrm{e}^{-\kappa x}\,\mathrm{e}^{\kappa\partial_x}]h_n(\sinh\kappa x|q) = 2q^{n/2}\cosh\kappa x\, h_n(\sinh\kappa x|q). \tag{4.7}$$

A glance at the Fourier transforms (2.9) and (3.14) reveals that we actually need q-difference equations for the functions $H_n(\sin\kappa x|q)\,\mathrm{e}^{-x^2/2}$ and $h_n(\sinh\kappa x|q)\,\mathrm{e}^{-x^2/2}$. So, one readily deduces from (4.1) and (4.7) that the desired equations can be written as (recall that $\widehat{p}_x = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ as before)

$$\cosh\kappa\widehat{p}_x\left[H_n(\sin\kappa x|q)\,\mathrm{e}^{-x^2/2}\right] = q^{-(2n+1)/4}\cos\kappa x\, H_n(\sin\kappa x|q)\,\mathrm{e}^{-x^2/2}, \tag{4.8}$$
$$\cos\kappa\widehat{p}_x\left[h_n(\sinh\kappa x|q)\,\mathrm{e}^{-x^2/2}\right] = q^{(2n+1)/4}\cosh\kappa x\, h_n(\sinh\kappa x|q)\,\mathrm{e}^{-x^2/2}, \tag{4.9}$$

respectively. These two q-difference equations are in fact interrelated by the similarity transformation by the unitary Fourier-transform operator $\widehat{\mathcal{F}}$, defined in (3.3). Indeed, to prove this assertion act on both sides of equation (4.8) by the operator $\widehat{\mathcal{F}}$ to obtain

$$\widehat{\mathcal{F}}\cosh\kappa\widehat{p}_x\left[H_n(\sin\kappa x|q)\,\mathrm{e}^{-x^2/2}\right] = q^{-(2n+1)/4}\widehat{\mathcal{F}}\cos\kappa x\, H_n(\sin\kappa x|q)\,\mathrm{e}^{-x^2/2}. \tag{4.10}$$

Substitute now the relation

$$H_n(\sin\kappa x|q)\,\mathrm{e}^{-x^2/2} = \mathrm{i}^n q^{n^2/4}\widehat{\mathcal{F}}^{-1}\left[h_n(\sinh\kappa x|q)\,\mathrm{e}^{-x^2/2}\right],$$

which follows from (3.14), into both sides of (4.10) and employ on the right side of it the identity

$$\widehat{\mathcal{F}}\cos\kappa x\widehat{\mathcal{F}}^{-1} = \cos\kappa\widehat{p}_x,$$

which is an easy consequence of (3.12). This yields

$$\widehat{\mathcal{F}}\cosh\kappa\widehat{p}_x\widehat{\mathcal{F}}^{-1}\left[h_n(\sinh\kappa x|q)\,\mathrm{e}^{-x^2/2}\right] = q^{-(2n+1)/4}\cos\kappa\widehat{p}_x\left[h_n(\sinh\kappa x|q)\,\mathrm{e}^{-x^2/2}\right]. \tag{4.11}$$

Comparison of (4.11) and (4.9) shows that these two q-difference equations coincide provided that the identity

$$\widehat{\mathcal{F}}\cosh\kappa\widehat{p}_x\widehat{\mathcal{F}}^{-1} = \cosh\kappa x \tag{4.12}$$

holds. But this is not hard to prove, bearing in mind the well-known series expansion

$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

The point is that the identity (3.9) from the previous section readily gets generalized (recall that $\widehat{\mathcal{F}}$ is a unitary operator) to

$$\widehat{\mathcal{F}}\widehat{p}_x^{2m}\widehat{\mathcal{F}}^{-1} = x^{2m}, \tag{4.13}$$

where m is an arbitrary integer number. So, after multiplying both sides of (4.13) by the factor $\kappa^{2m}/(2m)!$ and summing them with respect to m from zero to infinity, one arrives at the desired identity (4.12).

Similarly, one may start with the q-difference equation (4.9) and show that it is reducible to the q-difference equation (4.8), provided that

$$\widehat{\mathcal{F}}^{-1}\cosh\kappa x\widehat{\mathcal{F}} = \cosh\kappa\widehat{p}_x. \tag{4.14}$$

This identity follows at once from (4.12) (or can be deduced directly from (3.8) by a derivation similar to that of (4.12)), already proved above.

Thus, we have demonstrated that q-difference equations (4.8) and (4.9) are interrelated by the similarity transformation by the operator $\widehat{\mathcal{F}}$. This algebraic property of (4.8) and (4.9) explains the simple behavior of the q-Hermite polynomials (with $0 < q < 1$ and $1 < q < \infty$) with respect to the integral Fourier transform in (2.9). Obviously, it is more complicated than the mere commutativity of the operators $\widehat{N}$ and $\widehat{\mathcal{F}}$ in the case of the Hermite polynomials $H_n(x)$. But the curious fact is that although q-difference equations (4.8) and (4.9) are no longer invariant with respect to the interchange of the coordinate x and momentum operator $\widehat{p}_x := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$, those equations do exhibit simple behavior under this interchange $x \leftrightarrow \widehat{p}_x$. Indeed, since the interchange $x \leftrightarrow \widehat{p}_x$ entails the changes $\cos\kappa x \leftrightarrow \cos\kappa\widehat{p}_x$ and $\cosh\kappa x \leftrightarrow \cosh\kappa\widehat{p}_x$ in (4.8) and (4.9), this means that these equations convert into each other under the interchange $x \leftrightarrow \widehat{p}_x$. In other words, exactly as in the case of the ordinary Hermite polynomials $H_n(x)$, the similarity transformation of q-difference equations (4.8) and (4.9) by the operator $\widehat{\mathcal{F}}$ turns out to be equivalent to the interchange $x \leftrightarrow \widehat{p}_x$ in (4.8) and (4.9).

5. Concluding remarks

We have restricted our attention in the present work to only one q-extension of the classical Hermite polynomials $H_n(x)$, the continuous q-Hermite polynomials $H_n(x|q)$ with $0 < q < 1$ and $1 < q < \infty$. As mentioned in introduction, there are other similar pairs of q-analogs of $H_n(x)$, such as Rogers–Szegö and Stieltjes–Wigert families of q-polynomials [7], as well as the discrete q-Hermite polynomials of type I and type II [8]. Besides, it is known [7] that q-Laguerre polynomials and various q-exponential and q-Bessel functions also exhibit simple transformation properties with respect to the integral Fourier transform, similar to (2.9). Therefore it will be of interest to examine the possibility of implementing the same technique to the study of algebraic properties of those q-difference equations, which govern the above-mentioned instances of other q-extensions of classical special functions.

Acknowledgments

We are grateful to S L Woronowicz for a helpful discussion. We thank a referee for useful comments and suggestions, which have been taken into account in this final form of the paper. The participation of MKA in this work has been supported by the project No.25564 of CONACYT, whereas NMA has been supported by the UNAM-DGAPA project IN105008 'Óptica Matemática'.

References

[1] Macfarlane A J 1989 On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$ *J. Phys. A: Math. Gen.* **22** 4581–8

[2] Biedenharn L C 1989 The quantum group $SU_q(2)$ and a q-analogue of the Boson operators *J. Phys. A: Math. Gen.* **22** L873–8

[3] Rogers L J 1894 Second memoir on the expansion of certain infinite products *Proc. Lond. Math. Soc.* **25** 318–43

[4] Kulish P P and Damaskinsky E V 1990 On the q oscillator and the quantum algebra $su_q(1, 1)$ *J. Phys. A: Math. Gen.* **23** L415–9

[5] Atakishiyev N M and Suslov S K 1991 Difference analogs of the harmonic oscillator *Theor. Math. Phys.* **85** 1055–62

[6] Landau L D and Lifshitz E M 1991 *Quantum Mechanics (Non-relativistic Theory)* (Oxford: Pergamon)

[7] Atakishiyev N M 2000 *Fourier–Gauss Transforms of Some q-Special Functions (CRM Proceedings and Lecture Notes* vol 25*)* (Providence, RI: American Mathematical Society) pp 13–21

[8] Arik M, Atakishiyev N M and Rueda J P 2006 Discrete q-Hermite polynomials are linked by the integral and finite Fourier transforms *Int. J. Diff. Eqns.* **1** 195–204

[9] Mehta M L 1987 Eigenvalues and eigenvectors of the finite Fourier transform *J. Math. Phys.* **28** 781–5

[10] Matveev V B 2001 Intertwining relations between the Fourier transform and discrete Fourier transform, the related functional identities and beyond *Inverse Problems* **17** 633–57

[11] Atakishiyev N M 2006 On q-extensions of Mehta's eigenvectors of the finite Fourier transform *Int. J. Mod. Phys.* A **21** 4993–5006

[12] Atakishiyev N M, Rueda J P and Wolf K B 2007 On q-extended eigenvectors of the integral and finite Fourier transforms *J. Phys. A: Math. Theor.* **40** 12701–7

[13] McClellan J H and Parks T W 1972 Eigenvalue and eigenvector decomposition of the discrete Fourier transform *IEEE Trans. Audio Electroacoust.* **20** 66–74

[14] Dickinson B W and Steiglitz K 1982 Eigenvectors and functions of the discrete Fourier transform *IEEE Trans. Acoust. Speech Signal Process.* **30** 25–31

[15] Santhanam T S and Tekumalla A R 1976 Quantum mechanics in finite dimensions *Found. Phys.* **6** 583–9

[16] Grünbaum F A 1982 The eigenvectors of the discrete Fourier transform: a version of the Hermite functions *J. Math. Anal. Appl.* **88** 355–63

[17] Caola M J 1991 Self-Fourier functions *J. Phys. A: Math. Gen.* **24** L1143–4

[18] Horikis T P and McCallum M S 2006 Self-Fourier functions and self-Fourier operators *J. Opt. Soc. Am.* A **23** 829–34

[19] Gasper G and Rahman M 2004 *Basic Hypergeometric Functions* 2nd edn (Cambridge: Cambridge University Press)

[20] Andrews G E, Askey R and Roy R 1999 *Special Functions* (Cambridge: Cambridge University Press)

[21] Allaway Wm R 1980 Some properties of the q-Hermite polynomials *Can. J. Math.* **32** 686–94

[22] Askey R 1980 Continuous q-Hermite polynomials when $q > 1$ In *q-Series and Partitions (The IMA Volumes in Mathematics and Its Applications* vol 18*)* ed D Stanton (New York: Springer) pp 151–8

[23] Atakishiyev N M and Nagiyev Sh M 1994 On the wave functions of a covariant linear oscillator *Theor. Math. Phys.* **98** 162–6

[24] Atakishiyeva M K and Atakishiyev N M 2001 Fourier-Gauss transforms of bilinear generating functions for the continuous q-Hermite polynomials *Phys. At. Nucl.* **64** 2086–92

[25] Atakishiyev N M and Klimyk A U 2007 On factorization of q-difference equation for continuous q-Hermite polynomials *J. Phys. A: Math. Theor.* **40** 9311–7